Minmax Methods for Geodesics and Minimal Surfaces

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Lecture 1 : The Origin of Minmax,

Birkhoff Curve Shortening Process

and its Generalization to Surfaces

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$$\Longleftrightarrow \quad \nabla \partial_\theta \vec{\gamma} = 0 \quad \Longleftrightarrow \quad -\partial^2_{\theta^2} \vec{\gamma} + \partial_\theta (P^T_{\vec{\gamma}}) \partial_\theta \vec{\gamma} = 0$$

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Theorem [Hadamard 1898, Poincaré 1905, Cartan 1927]
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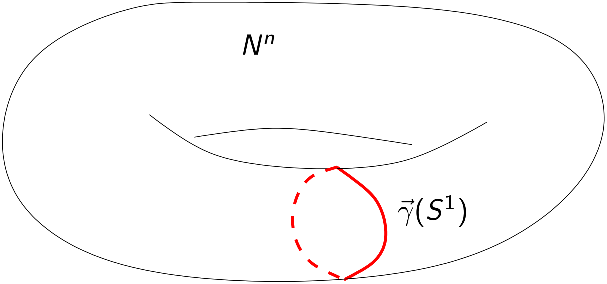

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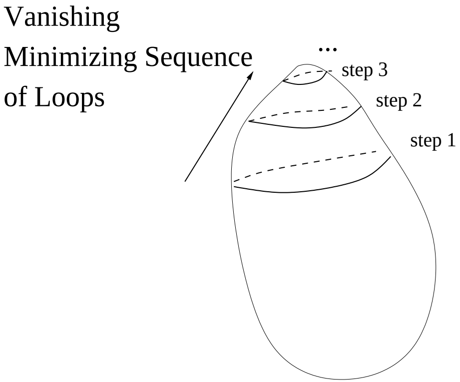

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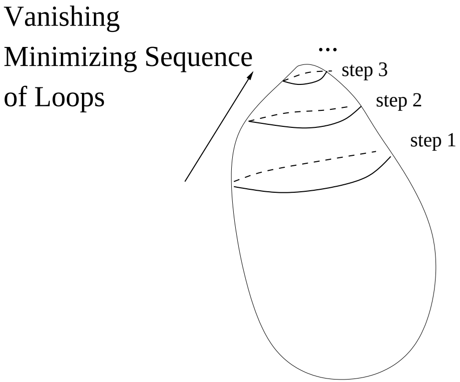

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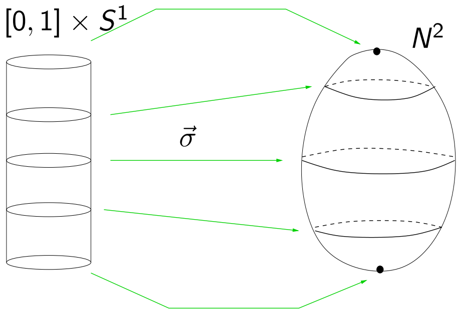

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where $p_k(t) \in C^0([0,1], N^n)$.Using shortest geodesic

$$\text{homotop } \vec{\sigma}(t, \cdot) \quad \text{to the constant map } p_k(t)$$

Observe

$$p_k([0,1]) \quad \text{is contractible.}$$

Hence $[\vec{\sigma}_k([0,1] \times S^1)] = 0$ in $H_2(N^2)$. □

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Observe

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$$G := \Lambda \cap \{\text{immersed closed geodesics}\}$$

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Proof : We "pull tight" a minimizing sequence $\vec{\sigma}_k$ into $\vec{\gamma}_k = \Psi(\vec{\sigma}_k)$. Hence

$$\forall\, \varepsilon > 0 \quad \exists\, \eta > 0 \quad \text{s.t. for } k \text{ large enough}$$

$$(2\pi)^{-1}\, L^2(\vec{\gamma}_k(t,\cdot)) = E(\vec{\gamma}_k(t,\cdot)) \geq W_{\vec{\sigma}_0} - \eta \implies \text{dist}(\vec{\gamma}_k(t,\cdot), G) \leq \varepsilon$$

□

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$$dist^2(\vec{\sigma}_1, \vec{\sigma}_2) \leq C \ [E(\vec{\sigma}_1) - E(\vec{\sigma}_2)]$$

where $C > 0$ only depends on N^n. □

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Lemma. Uniform $\epsilon-$regularity $\exists\, \epsilon_N > 0$ s.t. $\forall\ 0 \leq \sigma \leq 1$

$$\int_{B_r(x)} (1 + |du_\sigma|^2_{S^2})^{(1+\sigma)}\ dvol_{S^2} < \epsilon_N \quad \Longrightarrow \quad |du_\sigma|(x) \leq r^{-1}$$

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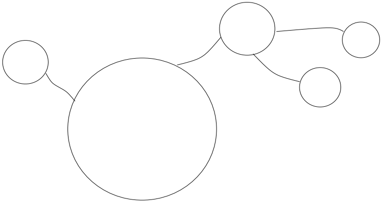

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The proof is very involved. Based on harmonic replacement method coming from the local convexity of the harmonic map Lagrangian.
It replaces Birkhoff curve shortening procedure.

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This has brought important results in codimension 1.

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Theorem [Marques, Neves, 2016] Let $3 \leq m \leq 7$ and (M^m, g) be a closed Riemannian manifold of positive Ricci curvature then there exists infinitely many embedded minimal surface of codimension 1 in M^m. □